

VELOCITY ABERRATION

by

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Introduction. The appendix to this paper contains diagrams and discussions of the basic optics of a two dimensional retroreflector for the various cases analyzed in this paper. This is included to provide an understanding of the physical processes described by the equations. The length of the cube corner from face to vertex is L . The width of the front face is $l = 2L$. The direction of the reflected beam can be calculated by computing the difference in optical path length for rays traveling parallel and anti-parallel to the direction of motion of the retroreflector.

1. Solid cube corner.

Let us consider a solid two-dimensional cube corner moving to the left with velocity v . The width of the front face is l and the index of refraction is n . The geometry is the same as that of figure 4 in the appendix for a hollow retroreflector. The time for a ray to travel from the right side of the cube to the left at velocity U_1 is T_1 . The time for a ray to travel from the left side to the right at velocity U_2 is T_2 .

The relativistic velocities U_1 and U_2 are given by the equations

$$U_1 = \frac{c + nv}{n + \frac{v}{c}} \quad (1.1)$$

and

$$U_2 = \frac{c - nv}{n - \frac{v}{c}} \quad (1.2)$$

where c is the velocity of light in vacuum. The distance traveled by the ray moving to the left is equal to the width l of the cube plus the distance moved by the cube in time T_1 at velocity v . The distance traveled by the ray moving to the right is equal to the width l of the cube minus the distance moved by the cube in time T_2 at velocity v . We have the two equations

$$U_1 T_1 = l + v T_1 \quad (1.3)$$

and

$$U_2 T_2 = l - v T_2 \quad (1.4)$$

Solving equation (1.3) for T_1 gives

$$(U_1 - v)T_1 = l$$

$$T_1 = \frac{l}{U_1 - v} \quad (1.5)$$

Solving equation (1.4) for T_2 gives

$$(U_2 + v)T_2 = l$$

$$T_2 = \frac{l}{U_2 + v} \quad (1.6)$$

If T_1 is greater than T_2 the ray traveling to the right will arrive first. It will be retroreflected and exit the cube corner ahead of the ray traveling to the left. As shown in figure 4 of the appendix, it will travel a distance $B'C = c(T_1 - T_2)$ before the ray traveling right exits the cube corner. The effect of this is to tilt the reflected wavefront to the left which we can define as a positive deflection angle. The velocity aberration angle α is given by dividing $B'C$ by $l = 2L$. This gives

$$\alpha = \frac{c}{l}(T_1 - T_2). \quad (1.7)$$

Substituting equations (1.5) and (1.6) into equation (1.7) gives

$$\alpha = \frac{c}{l} \left(\frac{l}{U_1 - v} - \frac{l}{U_2 + v} \right)$$

Canceling the factors of l gives

$$\alpha = c \left(\frac{1}{U_1 - v} - \frac{1}{U_2 + v} \right) \quad (1.8)$$

Evaluating the first denominator in equation (1.8) we have

$$U_1 - v = \frac{c + nv}{n + \frac{v}{c}} - v = \frac{c + nv - nv - \frac{v^2}{c}}{n + \frac{v}{c}} = \frac{c - \frac{v^2}{c}}{n + \frac{v}{c}} \quad (1.9)$$

$$= \frac{c^2 - v^2}{nc + v}$$

Evaluating the second denominator in equation (1.9) we have

$$\begin{aligned}
U_2 + v &= \frac{c - nv}{n - \frac{v}{c}} + v = \frac{c - nv + nv - \frac{v^2}{c}}{n - \frac{v}{c}} = \frac{c - \frac{v^2}{c}}{n - \frac{v}{c}} \\
&= \frac{c^2 - v^2}{nc - v}
\end{aligned} \tag{1.10}$$

Substituting equations (1.9) and (1.10) into equation (1.8) gives

$$\alpha = c \left(\frac{nc + v}{c^2 - v^2} - \frac{nc - v}{c^2 - v^2} \right) = c \frac{nc + v - nc + v}{c^2 \left(1 - \frac{v^2}{c^2} \right)}$$

The final result is

$$\alpha = \frac{2v}{c \left(1 - \frac{v^2}{c^2} \right)} \approx 2 \frac{v}{c} \tag{1.11}$$

The second order term in the denominator of equation (1.11) may be the result of the approximation used to define α in equation (1.7). The geometry is shown in Figure 4 of the appendix.

2. Hollow cube corner.

Since the index of refraction cancels in equation (1.11) the velocity aberration for a hollow cube corner should be exactly the same as for a solid cube corner. The equations have the same form up to equation (1.8). Since $n = 1$, the relativistic velocities in equations (1.1) and (1.2) reduce to

$$U_1 = U_2 = c \tag{2.1}$$

Substituting equation (2.1) into equation (1.8) gives

$$\alpha = c \left(\frac{1}{c - v} - \frac{1}{c + v} \right).$$

Combining the fractions we have

$$\alpha = c \left(\frac{c + v - c + v}{c^2 - v^2} \right).$$

This reduces to

$$\alpha = \frac{2v}{c \left(1 - \frac{v^2}{c^2} \right)}$$

which is the same expression obtained for a solid cube corner.

3. Stationary cube corner with moving fluid.

This is essentially the Fizeau experiment where mirrors are used to pass light through a pipe of length l containing water of index of refraction n moving at velocity v from right to left. The geometry is shown in figure 5 of the appendix. This analysis is a modified version of a calculation done by T.P Startsev (private communication). In place of equations (1.5) and (1.6) we have

$$T_1 = \frac{l}{U_1} \quad (3.1)$$

and

$$T_2 = \frac{l}{U_2} \quad (3.2)$$

Substituting (3.1) and (3.2) into equation (1.7) gives

$$\begin{aligned} \alpha &= \frac{c}{l} \left(\frac{l}{U_1} - \frac{l}{U_2} \right) \\ &= c \left(\frac{1}{U_1} - \frac{1}{U_2} \right) \end{aligned} \quad (3.3)$$

Substituting equations (1.1) and (1.2) into the expression in parenthesis in equation (3.3) we have

$$\begin{aligned} \frac{1}{U_1} - \frac{1}{U_2} &= \frac{n + \frac{v}{c}}{c + nv} - \frac{n - \frac{v}{c}}{c - nv} \\ &= \frac{\left(n + \frac{v}{c}\right)(c - nv) - \left(n - \frac{v}{c}\right)(c + nv)}{c^2 - n^2v^2} \end{aligned}$$

Neglecting the second term in the denominator and multiplying the terms in the numerator we have

$$\begin{aligned} \frac{1}{U_1} - \frac{1}{U_2} &\approx \frac{1}{c^2} \left(\left[nc + v - n^2v - \frac{nv^2}{c} \right] - \left[nc - v + n^2v - \frac{nv^2}{c} \right] \right) \\ &= \frac{1}{c^2} (2v - 2n^2v) \end{aligned}$$

The final result is

$$\frac{1}{U_1} - \frac{1}{U_2} = \frac{2v}{c^2} (1 - n^2) \quad (3.4)$$

Substituting equation (3.4) into equation (3.3) gives

$$\alpha = \frac{2v}{c}(1 - n^2) \quad (3.5)$$

This expression is negative. As anticipated in figure 5, the return beam is deflected to the right which is in the opposite direction from the direction of motion of the fluid.

4. Moving cube corner with stationary fluid.

This case may not be very realistic physically but it is presented as a gedanken experiment to show the contributions to the total velocity aberration. The relativistic velocities are

$$U_1 = U_2 = \frac{c}{n} \quad (4.1)$$

If we substitute equation (4.1) into equation (1.8), we have

$$\alpha = c \left(\frac{1}{\frac{c}{n} - v} - \frac{1}{\frac{c}{n} + v} \right)$$

Evaluation the fractions gives

$$\alpha = c \left(\frac{n}{c - nv} - \frac{n}{c + nv} \right)$$

Combining the fractions gives

$$\alpha = c \left(\frac{nc + n^2v - nc + n^2v}{c^2 - n^2v^2} \right)$$

Neglecting the second order term in the denominator we have

$$\alpha = n^2 \frac{2v}{c} \quad (4.2)$$

Suppose one did not know about the Fizeau effect or the equation for the addition of velocities from special relativity. Equation (4.2) is the result that would be obtained by assuming that the velocity of light in a moving fluid is given by equation (4.1) instead of equations (1.1) and (1.2). This result is greater than the usual expression $2v/c$. Equation (4.2) could be rewritten as

$$\alpha = \frac{2v}{c} + \frac{2v}{c}(n^2 - 1) \quad (4.3)$$

The second term is equal and opposite to equation (3.5) derived in section 3.

5. Moving fluid and moving cube corner.

For a cube corner in orbit, both the mirrors and the dielectric are moving together at velocity v . Equation (4.3) is equal and opposite to equation (3.5). If we add equations (3.5) derived with only the fluid moving and (4.3) derived with only the mirrors moving, the index of refraction cancels leaving

$$\alpha = 2 \frac{v}{c} \quad (5.1)$$

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Reference

Effect of motion of the optical medium in optical location

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Appendix

Basic Retroreflector Optics

Section A below shows the motion of a light wave through a two-dimensional hollow retroreflector both as a ray and as a wave. Section B considers the case of a moving hollow retroreflector. Section C discusses the effect of a moving fluid in a stationary hollow retroreflector. Section D discussed the case of a hollow retroreflector moving through a stationary fluid.

A. Stationary Hollow Retroreflector

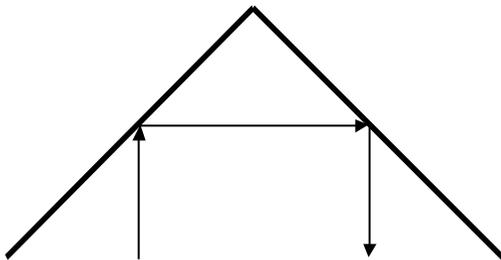


Figure 1a. Ray entering the left half.

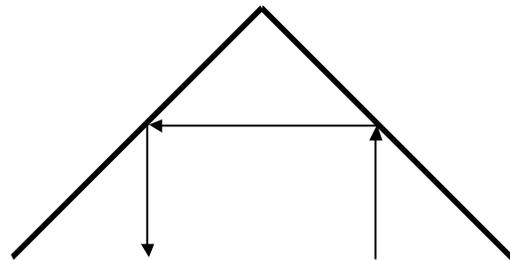


Figure 1b. Ray entering the right half.

Figure 1 shows a hollow two-dimensional retroreflector. The angle between the faces is 90 degrees. In figure 1a a ray traveling up enters the left half of the retroreflector, strikes the left face at a 45 degree angle, is reflected through a 90 degree angle, and ends up traveling to

the right. It strikes the right face, is reflected by 90 degrees, and ends up traveling down out of the retroreflector. Figure 1b shows a ray entering the right half and being retroreflected in a similar manner.

A two-dimensional retroreflector has two sectors corresponding to the two possible orders of reflection. In a three dimensional retroreflector there are 6 sectors corresponding to the 6 possible orders of reflection from the three back faces.

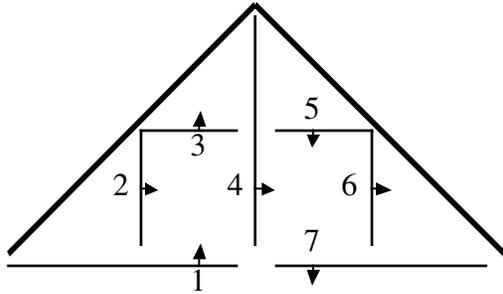


Figure 2a. Wave entering left half

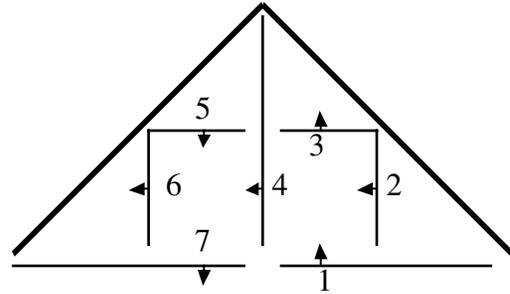


Figure 2b. Wave entering right half

Figure 2 shows the details of what happens to the waves incident on the two sectors. In Figure 2a, the wave enters the left half traveling up as shown by the arrow on wavefront number 1. After traveling halfway into the retroreflector section 2 has been reflected from the left face and is traveling to the right. Section 3 is still traveling up. When the wave reaches the vertex it has been completely reflected from the left face and is traveling to the right. Halfway back out of the retroreflector section 5 has been reflected from the right face and is traveling down. Section 6 is still traveling to the right. Section 7 shows the wave exiting the retroreflector traveling down.

Figure 2b shows the wave entering the right half and exiting the left half. Both the wavefronts have been reversed left to right in the process of being retroreflected.

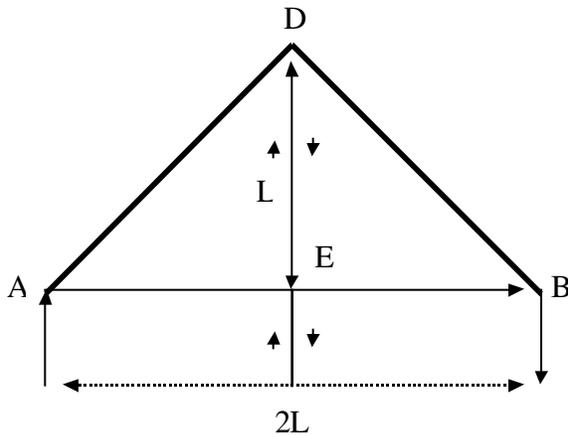


Figure 3a. Ray tracing for left sector.

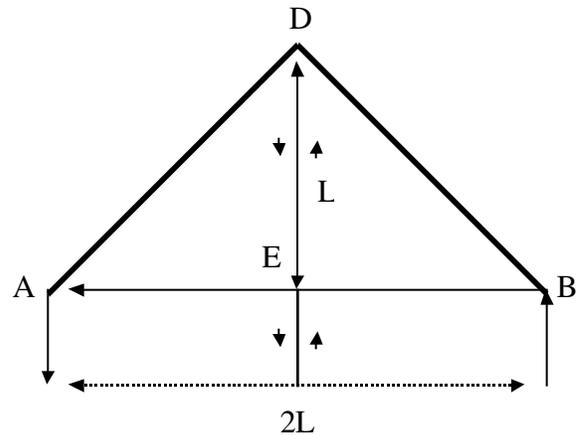


Figure 3b. Ray tracing for right sector.

Figure 3 shows a ray tracing for the two sectors. In Figure 3a the ray entering at point A travels to the right and exits at point B. The ray entering at point E travels to the vertex at point D and then back to point E. In Figure 3b, the ray entering at point B travels to the left

and exits at point A . The ray entering at point E travels to the vertex at point D and then back to point E . The ray entering the center at point E is common to both sectors.

The ray entering at point A travels a distance $2L$ to point B . The ray entering at point E travels a distance $2L$ in going to point D and back to E . The ray entering at point B travels a distance $2L$ to point A . Since the path length is the same for all rays, the exiting wave has a flat phase front and is traveling back along the same direction that it entered.

B. Moving hollow retroreflector.

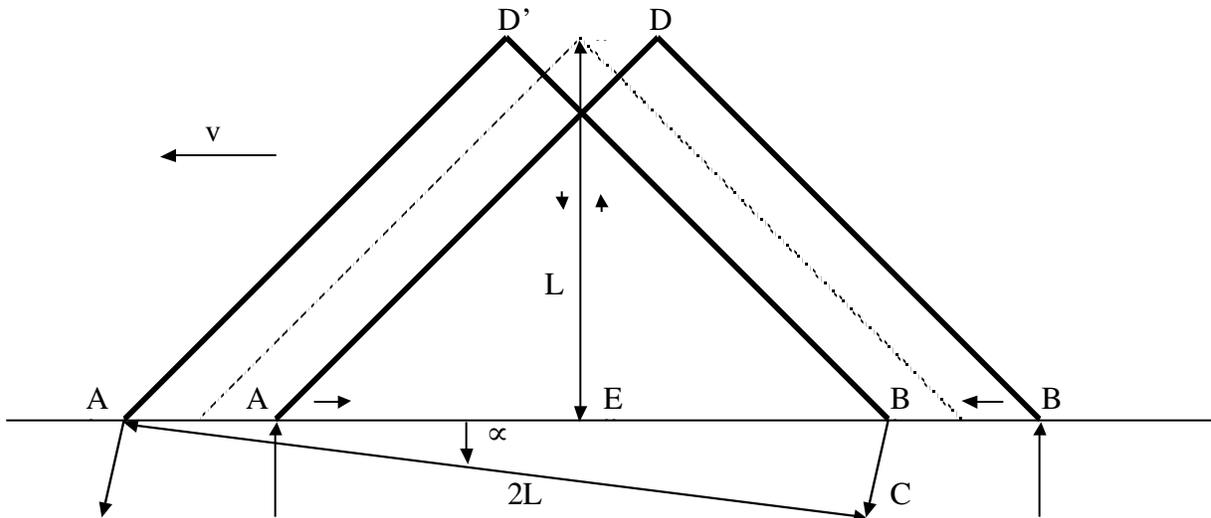


Figure 4. Moving retroreflector

Figure 4 shows a two dimensional retroreflector moving at velocity v to the left. In this section we will consider the retroreflector to be hollow. The vertex is at point D as the wave enters the retroreflector. The wave reaches the vertex at point D' . As it leaves the retroreflector the vertex is at point D' . The length from the front face to the vertex is L .

The time required to move the distance L from the front face to the vertex at velocity c is L/c . In that time the retroreflector moves a distance $v(L/c)$. By the time the ray has returned to the front face, the retroreflector has moved a total distance

$$AA' = 2Lv/c.$$

Because of the motion of the retroreflector, the ray traveling to the right exits the cube corner before the ray traveling to the vertex. The ray traveling to the left exits the cube corner after the ray traveling to the vertex. The effect of the motion of the cube corner is to tilt the direction of the reflected wave by the angle α .

The angle α can be calculated approximately as follows. The optical path length for the ray traveling to the left is

$$2L + 2Lv/c$$

The optical path length for the ray traveling to the right is

$$2L - 2Lv/c$$

the difference in optical path length is

$$4Lv/c$$

Dividing the difference in optical path length by $2L$ which is the width of the front face gives

$$\alpha = 2v/c$$

C. Stationary hollow retroreflector with moving fluid

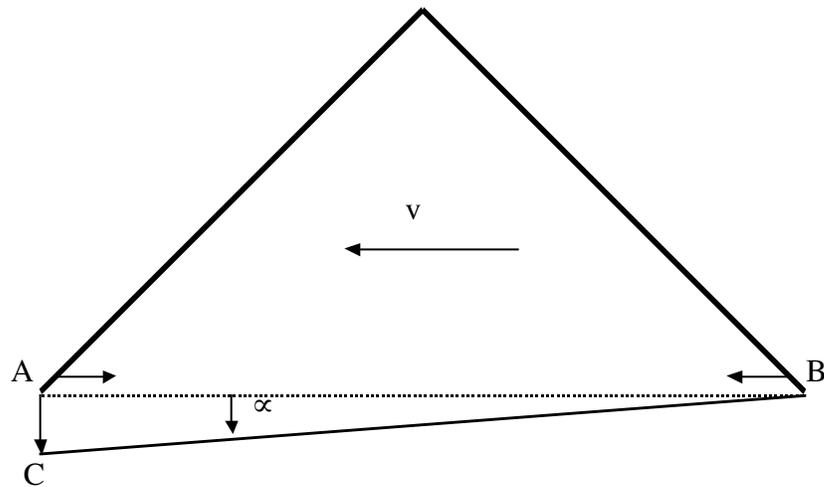


Figure 5. Stationary hollow retroreflector with fluid moving from right to left

Figure 5 shows a stationary two dimensional retroreflector with a fluid of index of refraction n flowing from right to left between the mirrors at velocity v . The velocity of light in a stationary fluid is c/n . As a result of the Fizeau effect, a moving fluid “drags” the light with a fraction of its own velocity. The velocity of the ray which enters at B and travels to the left is slightly greater than c/n . The velocity of the ray which enters at A and travels to the right is slightly less than c/n because it is traveling “against the current”. The ray traveling to the left arrives first and travels to C by the time the ray traveling to the right reaches B. The result is that the reflected wave is deflected to the right. This is in the opposite direction from the velocity of the fluid which flows from right to left. The angle of deflection has been derived in section 3 and is given by equation (3.5)

D. Moving retroreflector with stationary fluid.

The velocity aberration for this case can be calculated approximately using figure 4. The time required to move the distance L from the front face to the vertex at velocity (c/n) is $L/(c/n) = nL/c$. In that time the retroreflector moves a distance $v(nL/c)$. By the time the ray has returned to the front face, the retroreflector has moved a total distance

$$AA' = 2nLv/c.$$

The optical path length for the ray traveling to the left is

$$n(2L + 2nLv/c)$$

The optical path length for the ray traveling to the right is

$$n(2L - 2nLv/c)$$

the difference in optical path length is

$$4n^2Lv/c$$

Dividing the difference in optical path length by $2L$ which is the width of the front face gives

$$\alpha = n^2 \frac{2v}{c}$$

This is the same as equation (4.2) derived in a more rigorous manner in section 4.